

# On the spectra of the quantized action-variables of the compactified Ruijsenaars-Schneider system

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## Abstract

A simple derivation of the spectra of the action-variables of the quantized compactified Ruijsenaars-Schneider system is presented. The spectra are obtained by combining Kähler quantization with the identification of the classical action-variables as a standard toric moment map on the complex projective space. The result is consistent with the Schrödinger quantization of the system worked out previously by van Diejen and Vinet.

# 1 Introduction

The definition of the compactified Ruijsenaars-Schneider system [9] begins with the local Hamiltonian<sup>1</sup>

$$H^{\text{loc}}(x, p) \equiv \sum_{j=1}^n \cos p_j \prod_{k \neq j}^n \left[ 1 - \frac{\sin^2 y}{\sin^2 \frac{a(x_j - x_k)}{2}} \right]^{\frac{1}{2}}. \quad (1.1)$$

The variables  $\delta_j = e^{iax_j}$  ( $j = 1, \dots, n$ ) are interpreted as the positions of  $n$  “particles” moving on the circle and the canonically conjugate momenta  $p_j$  encode the compact variables  $\Theta_j = e^{-ip_j}$ . The parameters  $a$  and  $y$  play the role of coupling constants. Here, the center of mass condition  $\prod_{j=1}^n \delta_j = \prod_{j=1}^n \Theta_j = 1$  is also adopted. Denoting the standard maximal torus of  $SU(n)$  as  $S\mathbb{T}_n$ , the local phase space is

$$M_y^{\text{loc}} \equiv \{(\delta, \Theta) \mid \delta = (\delta_1, \dots, \delta_n) \in \mathcal{D}_y, \Theta = (\Theta_1, \dots, \Theta_n) \in S\mathbb{T}_n\}, \quad (1.2)$$

where the domain  $\mathcal{D}_y \subset S\mathbb{T}_n$  is chosen in such a way to guarantee that  $H^{\text{loc}}$  takes real values. The non-emptiness of  $\mathcal{D}_y$  is ensured by the restriction  $|y| < \frac{\pi}{n}$ . The symplectic form on  $M_y^{\text{loc}}$  is defined by

$$\Omega_a^{\text{loc}} \equiv \frac{1}{a} \text{tr} (\delta^{-1} d\delta \wedge \Theta^{-1} d\Theta) = \sum_{j=1}^n dx_j \wedge dp_j. \quad (1.3)$$

The Hamiltonian  $H^{\text{loc}}$  can be recast as the real part of the trace of the unitary Lax matrix  $L_y^{\text{loc}}$ :

$$L_y^{\text{loc}}(\delta, \Theta)_{jl} \equiv \frac{e^{iy} - e^{-iy}}{e^{iy} \delta_j \delta_l^{-1} - e^{-iy}} W_j(\delta, y) W_l(\delta, -y) \Theta_l \quad (1.4)$$

with the positive functions

$$W_j(\delta, y) := \prod_{k \neq j}^n \left[ \frac{e^{iy} \delta_j - e^{-iy} \delta_k}{\delta_j - \delta_k} \right]^{\frac{1}{2}}. \quad (1.5)$$

The flows generated by the spectral invariants of  $L_y^{\text{loc}}$  commute, but are not complete on  $M_y^{\text{loc}}$ . Ruijsenaars [9] has shown that one can realize  $(M_y^{\text{loc}}, \Omega_a^{\text{loc}})$  as a dense open submanifold of the complex projective space  $\mathbb{C}P(n-1)$  equipped with a multiple of the Fubini-Study symplectic form, and thereby the commuting local flows generated by  $L_y^{\text{loc}}$  extend to complete Hamiltonian flows on the compact phase space  $\mathbb{C}P(n-1)$ . Ruijsenaars himself referred to the extended system on  $\mathbb{C}P(n-1)$  with complete flows as to the compactified III<sub>b</sub> system.

A rather complete solution of the quantized compactified Ruijsenaars-Schneider system was obtained by van Diejen and Vinet [10] by means of “Schrödinger quantization”, that is by explicit diagonalization of the commuting Hamiltonian operators in a coordinate representation. Here, our main purpose is to give an alternative, very simple, derivation of the result of [10] regarding the joint spectrum of the action-variables. We note that our parameter  $a > 0$  corresponds to  $\alpha$  in [10] and the constants  $g, M$  that appear in [10] are related to our parameters  $y, a$  above by

$$g \equiv \frac{2|y|}{a}, \quad M \equiv \frac{2}{a}(\pi - n|y|). \quad (1.6)$$

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<sup>1</sup>The index  $k$  in the next product  $\prod_{k \neq j}^n$  runs over  $\{1, 2, \dots, n\} \setminus \{j\}$ , and similar notation is used throughout.

We follow the convention of [5] in the present paper, except that in this reference the constant  $a$  was set equal to 2.

Classically, the action-variables can be identified with the components of the moment map of a standard Hamiltonian  $\mathbb{T}_{n-1}$  torus action on  $(\mathbb{C}P(n-1), M\omega_{\text{FS}})$ , and we shall obtain the quantized spectra by combining this identification with the Kähler quantization of the projective space. The identification just mentioned relies on a very non-trivial symplectic automorphism of  $\mathbb{C}P(n-1)$  that encodes the self-duality of the system discovered by Ruijsenaars [9]. In our recent paper [5], we found a geometric interpretation of the self-duality. Namely, we have shown that the compactified Ruijsenaars-Schneider system can be obtained by an appropriate quasi-Hamiltonian reduction of the internally fused quasi-Hamiltonian double<sup>2</sup>  $SU(n) \times SU(n)$ , and the self-duality is due to the democracy enjoyed by the two  $SU(n)$  factors of the double. More specifically, we demonstrated that the reduced phase space is a Hamiltonian toric manifold and that its identification with  $\mathbb{C}P(n-1)$  follows from the Delzant theorem of symplectic topology. Next we recall the notion of quasi-Hamiltonian reduction [1] and then review the pertinent results of [5]. Finally we present the derivation of the quantized spectra, which is an easy by-product of the geometric picture.

## 2 Quasi-Hamiltonian reduction

Consider a Lie group  $G$  acting on a manifold  $D$ . For a  $G$ -invariant *symplectic* form  $\omega$  on  $D$  the corresponding Poisson bracket  $\{\alpha, \beta\}$  of  $G$ -invariant functions  $\alpha, \beta$  is again  $G$ -invariant. The quotient  $D/G$  is thus naturally equipped with a Poisson structure. The choice of a symplectic leaf of this Poisson structure on  $D/G$  is known as a symplectic reduction of  $D$ .

Alekseev, Malkin and Meinrenken have shown that a  $G$ -invariant 2-form  $\omega$  on a manifold  $D$  obeying the axioms of quasi-Hamiltonian geometry *need not be symplectic* and still induces a Poisson structure on  $D/G$  [1]. This happens because the axioms of quasi-Hamiltonian geometry imply that for each  $G$ -invariant function  $\alpha$  there exists a unique  $G$ -invariant vector field  $v_\alpha$  on  $D$  verifying

$$\omega(v_\alpha, \cdot) = d\alpha, \quad (2.1)$$

and, moreover, the function

$$\{\alpha, \beta\} := \omega(v_\alpha, v_\beta) \quad (2.2)$$

is  $G$ -invariant and it defines a Poisson bracket on the space of the  $G$ -invariant functions on  $D$ . The choice of a symplectic leaf of this Poisson structure on  $D/G$  is called a quasi-Hamiltonian reduction of  $D$ .

Of course  $D/G$  is not a smooth manifold in general, but nevertheless the above remarks convey the main idea of reduction. Below we recall the axioms of quasi-Hamiltonian geometry and the reduction procedure in a more precise manner.

Let  $G$  be a compact Lie group with Lie algebra  $\mathcal{G}$ . Fix an invariant scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{G}$  and denote by  $\vartheta$  and  $\bar{\vartheta}$ , respectively, the left- and right-invariant Maurer-Cartan forms on  $G$ . For a  $G$ -manifold  $D$  with action  $\Psi : G \times D \rightarrow D$ , we use  $\Psi_\eta(x) := \Psi(\eta, x)$  and let  $\zeta_D$  denote

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<sup>2</sup>Via the connection between Chern-Simons models and quasi-Hamiltonian geometry [1], this confirms the related conjectures of Gorsky and Nekrasov [6].

the vector field on  $D$  that corresponds to  $\zeta \in \mathcal{G}$ . The adjoint action of  $G$  on itself is given by  $\text{Ad}_\eta(\tilde{\eta}) := \eta\tilde{\eta}\eta^{-1}$ , and  $\text{Ad}_\eta$  denotes also the induced action on  $\mathcal{G}$ .

By definition [1], a quasi-Hamiltonian  $G$ -space  $(D, G, \omega, \mu)$  is a  $G$ -manifold  $D$  equipped with an invariant 2-form  $\omega \in \Lambda^2(D)^G$  and with an equivariant map  $\mu : D \rightarrow G$ ,  $\mu \circ \Psi_\eta = \text{Ad}_\eta \circ \mu$ , in such way that the following conditions hold.

(a1) The differential of  $\omega$  is given by

$$d\omega = -\frac{1}{12}\mu^*\langle\vartheta, [\vartheta, \vartheta]\rangle. \quad (2.3)$$

(a2) The infinitesimal action is related to  $\mu$  and  $\omega$  by

$$\omega(\zeta_D, \cdot) = \frac{1}{2}\mu^*\langle\vartheta + \bar{\vartheta}, \zeta\rangle, \quad \forall \zeta \in \mathcal{G}. \quad (2.4)$$

(a3) At each  $x \in D$ , the kernel of  $\omega_x$  is provided by

$$\text{Ker}(\omega_x) = \{\zeta_D(x) \mid \zeta \in \text{Ker}(\text{Ad}_{\mu(x)} + \text{Id}_{\mathcal{G}})\}. \quad (2.5)$$

The map  $\mu$  is called the moment map.

Any  $G$ -invariant function  $h$  on  $D$  induces a unique vector field  $v_h$  on  $D$  that satisfies (2.1) and preserves  $\mu$  as well as  $\omega$ . A quasi-Hamiltonian dynamical system  $(D, G, \omega, \mu, h)$  is a quasi-Hamiltonian  $G$ -space with a distinguished  $G$ -invariant function  $h \in C^\infty(D)^G$ , the Hamiltonian.

The quasi-Hamiltonian reduction of a quasi-Hamiltonian dynamical system  $(D, G, \omega, \mu, h)$  can be determined by choosing an element  $\mu_0 \in G$ . We say that  $\mu_0$  is *strongly regular* if it satisfies the following two conditions:

1. The subset  $\mu^{-1}(\mu_0) := \{x \in D \mid \mu(x) = \mu_0\}$  is an embedded submanifold of  $D$ .
2. If  $G_0 < G$  is the isotropy group of  $\mu_0$  with respect to the adjoint action, then the quotient  $\mu^{-1}(\mu_0)/G_0$  is a manifold for which the canonical projection  $p : \mu^{-1}(\mu_0) \rightarrow \mu^{-1}(\mu_0)/G_0$  is a smooth submersion.

The result of the reduction based on a strongly regular element  $\mu_0$  is a standard Hamiltonian system,  $(P, \hat{\omega}, \hat{h})$ . The reduced phase space  $P$  is the manifold

$$P \equiv \mu^{-1}(\mu_0)/G_0, \quad (2.6)$$

which carries the reduced symplectic form  $\hat{\omega}$  and reduced Hamiltonian  $\hat{h}$  uniquely defined by

$$p^*\hat{\omega} = \iota^*\omega, \quad p^*\hat{h} = \iota^*h, \quad (2.7)$$

where  $\iota : \mu^{-1}(\mu_0) \rightarrow D$  is the tautological embedding.

We stress that  $\hat{\omega}$  is a symplectic form in the usual sense, whilst  $\omega$  is neither closed nor globally non-degenerate in general. The Hamiltonian vector field and the flow defined by  $\hat{h}$  on  $P$  can be obtained by first restricting the quasi-Hamiltonian vector field  $v_h$  and its flow to the “constraint surface”  $\mu^{-1}(\mu_0)$  and then applying the canonical projection  $p$ .

We wish to note for clarity that the group valued quasi-Hamiltonian moment map utilized here is different from the group valued Poisson-Lie moment map introduced by Lu [8]. We used the Poisson-Lie moment map in a previous paper [4] to describe the standard trigonometric Ruijsenaars-Schneider system and its dual [9], which are different from the system (1.1).

### 3 The reduction of the internally fused double

Let  $G \equiv SU(n)$ . The quasi-Hamiltonian manifold  $(D, G, \omega, \mu)$  that we are going to reduce is provided by the Cartesian product

$$D := G \times G = \{(A, B) \mid A, B \in G\}. \quad (3.1)$$

The invariant scalar product on  $\text{Lie}(G) = \mathfrak{su}(n)$  is given by

$$\langle \zeta, \tilde{\zeta} \rangle := -\frac{1}{a} \text{tr}(\zeta \tilde{\zeta}), \quad \forall \zeta, \tilde{\zeta} \in \mathfrak{su}(n). \quad (3.2)$$

The group  $G$  acts on  $D$  by componentwise conjugation

$$\Psi_\eta(A, B) := (\eta A \eta^{-1}, \eta B \eta^{-1}). \quad (3.3)$$

The 2-form  $\omega$  on  $D$  reads

$$\omega := \frac{1}{a} \langle A^{-1} dA \frown dBB^{-1} \rangle + \frac{1}{a} \langle dAA^{-1} \frown B^{-1} dB \rangle - \frac{1}{a} \langle (AB)^{-1} d(AB) \frown (BA)^{-1} d(BA) \rangle, \quad (3.4)$$

and the  $G$ -valued moment map  $\mu$  is defined by

$$\mu(A, B) = ABA^{-1}B^{-1}. \quad (3.5)$$

In order to define a quasi-Hamiltonian dynamical system on the “internally fused double”  $D$  described above [1] we need also a  $G$ -invariant Hamiltonian. We shall in fact consider two families of such Hamiltonians each one containing  $(n-1)$  members. They are given by the so-called *spectral functions* on each  $SU(n)$  factor of the double.

Roughly speaking, the spectral functions on  $SU(n)$  evaluated at a point  $C \in SU(n)$  are defined as logarithms of ratios of two neighbouring eigenvalues of  $C$ . More precisely, we define the alcove  $\mathcal{A}$  by

$$\mathcal{A} := \left\{ (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \mid \xi_j \geq 0, \quad j = 1, \dots, n, \quad \sum_{j=1}^n \xi_j = \pi \right\} \quad (3.6)$$

and consider the injective map  $\delta$  from  $\mathcal{A}$  into the subgroup  $S\mathbb{T}_n$  of the diagonal elements of  $SU(n)$  given by

$$\delta_{11}(\xi) := e^{\frac{2i}{n} \sum_{j=1}^n j \xi_j}, \quad \delta_{kk}(\xi) := e^{2i \sum_{j=1}^{k-1} \xi_j} \delta_{11}(\xi), \quad k = 2, \dots, n. \quad (3.7)$$

With the aid of the fundamental weights  $\Lambda_k$  of  $\mathfrak{su}(n)$  represented by the diagonal matrices  $\Lambda_k \equiv \sum_{j=1}^k E_{jj} - \frac{k}{n} \mathbf{1}_n$ , the matrix  $\delta(\xi)$  can be written in the form

$$\delta(\xi) = \exp \left( -2i \sum_{k=1}^{n-1} \xi_k \Lambda_k \right). \quad (3.8)$$

Every element  $C \in SU(n)$  can be diagonalized as

$$C = \eta^{-1} \delta(\xi) \eta, \quad (3.9)$$

for some  $\eta \in SU(n)$  and unique  $\xi \in \mathcal{A}$ . By definition, the  $j^{\text{th}}$  component  $\xi_j$  ( $j = 1, \dots, n$ ) of the alcove element  $\xi$  entering the decomposition (3.9) is the value of the spectral function  $\xi_j$  on  $C \in SU(n)$ .

Consider now the  $2(n-1)$  distinguished  $G$ -invariant Hamiltonians  $\alpha_j, \beta_j$  on the double  $D$  defined in terms of the spectral functions according to

$$\alpha_j(A, B) := \frac{2}{a}\xi_j(A), \quad \beta_j(A, B) := \frac{2}{a}\xi_j(B), \quad j = 1, \dots, n-1. \quad (3.10)$$

It turns out that for each  $j$  the associated quasi-Hamiltonian vector fields  $v_{\alpha_j}$  can be integrated to a very simple circle action on  $D$ :

$$(A, B\eta(A)^{-1}\text{diag}(1, 1, \dots, 1, e^{it}, e^{-it}, 1, \dots, 1)\eta(A)), \quad t \in \mathbb{R}. \quad (3.11)$$

Here the phase  $e^{it}$  sits in the  $j^{\text{th}}$  entry of the diagonal and  $\eta(A)$  is given by the diagonalization  $A = \eta(A)^{-1}\delta(\xi)\eta(A)$ . Similarly, the following  $2\pi$ -periodic curve in  $D$  is an integral curve of the vector field  $v_{\beta_j}$ :

$$(A\eta(B)^{-1}\text{diag}(1, 1, \dots, 1, e^{-it}, e^{it}, 1, \dots, 1)\eta(B), B), \quad t \in \mathbb{R}. \quad (3.12)$$

When taken together for every  $j = 1, \dots, n-1$ , the flows (3.11) give rise to the  $\alpha$ -generated torus action, and the flows (3.12) yield the  $\beta$ -generated torus action on  $D$ . We note in passing that the spectral Hamiltonians  $\alpha_j$  and  $\beta_j$  are smooth only on the dense open subset  $D_{\text{reg}} \subset D$  where  $A$  and  $B$  have distinct eigenvalues, but this does not lead to any difficulty since the constraint surface of our reduction turns out to be a submanifold of  $D_{\text{reg}}$ .

Now we state two theorems proved in [5] that characterize our reduction of the quasi-Hamiltonian dynamical systems based on the spectral Hamiltonians  $\alpha_j$  and  $\beta_j$  on the double.

**Theorem 1.** *The choice  $\mu_0 = \text{diag}(e^{2iy}, \dots, e^{2iy}, e^{2(1-n)iy})$  defines a strongly regular value of the moment map  $\mu$  (3.5) whenever the real parameter  $y$  verifies the condition  $0 < |y| < \frac{\pi}{n}$ . The corresponding reduced phase space  $P$  (2.6) is a smooth, compact manifold of dimension  $2(n-1)$ .*

**Theorem 2.** *The reduced phase space  $P = \mu^{-1}(\mu_0)/G_0$  is connected. Moreover, both the  $(\alpha_1, \dots, \alpha_{n-1})$ -generated and the  $(\beta_1, \dots, \beta_{n-1})$ -generated torus actions on the double descend to the reduced phase space  $P$ , where they become Hamiltonian and effective.*

Let us recall that a *Hamiltonian toric manifold*<sup>3</sup> is a compact connected symplectic manifold of dimension  $2(n-1)$  equipped with an effective Hamiltonian action of a torus of dimension  $(n-1)$ . Theorems 1 and 2 ensure that the reduced phase space  $(P, \hat{\omega})$  (2.7) is a Hamiltonian toric manifold in two different ways (i.e.  $\alpha$ -generated way and  $\beta$ -generated way).

The next theorem, proved again in [5], gives the key for the identification of the reduced phase space with the complex projective space  $\mathbb{CP}(n-1)$ .

**Theorem 3.** *The common image of the reduced phase space  $P$  under both  $(n-1)$ -tuples of reduced spectral Hamiltonians  $(\hat{\alpha}_1, \dots, \hat{\alpha}_{n-1})$  and  $(\hat{\beta}_1, \dots, \hat{\beta}_{n-1})$  is the convex polytope  $\frac{2}{a}\mathcal{P}_y$ , where*

$$\mathcal{P}_y := \left\{ (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \mid \xi_j \geq |y|, \quad j = 1, \dots, n-1, \quad \sum_{j=1}^{n-1} \xi_j \leq \pi - |y| \right\}. \quad (3.13)$$

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<sup>3</sup>A review of these compact completely integrable systems can be found in [2].

In order to be able to put Theorem 3 to use, and also for reference in Section 4, we need to sketch an auxiliary symplectic reduction treatment of  $\mathbb{CP}(n-1)$ . For this, take the symplectic vector space  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$  endowed with the Darboux form  $\Omega = i \sum_{k=1}^n d\bar{u}_k \wedge du_k$ , where  $u_k$  are the components of the vector  $u$  that runs over  $\mathbb{C}^n$ . Then consider the Hamiltonian action  $\psi$  of the group  $U(1)$  on  $\mathbb{C}^n$  operating as  $\psi_{e^{i\rho}}(u) := e^{i\rho}u$ . This  $U(1)$  action is generated by the moment map  $\chi : \mathbb{C}^n \rightarrow \mathbb{R}$ ,

$$\chi(u) \equiv \sum_{k=1}^n J_k \quad \text{with} \quad J_k := |u_k|^2 \quad (\forall k = 1, \dots, n). \quad (3.14)$$

For any fixed value  $M > 0$ , ordinary (Marsden-Weinstein) symplectic reduction of  $(\mathbb{C}^n, \Omega)$  yields the reduced phase space

$$\chi^{-1}(M)/U(1) \equiv \mathbb{CP}(n-1). \quad (3.15)$$

The corresponding reduced symplectic form is  $M\omega_{\text{FS}}$ , where  $\omega_{\text{FS}}$  is the standard Fubini-Study form of  $\mathbb{CP}(n-1)$ . The functions  $J_k$  are  $U(1)$  invariant and thus descend to smooth functions on the reduced phase space  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$ , which we shall denote below as  $\hat{J}_k$ .

Now focus on the action  $R : \mathbb{T}_{n-1} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  of the torus  $\mathbb{T}_{n-1}$  on  $\mathbb{C}^n$  furnished by

$$R_\tau(u_1, \dots, u_{n-1}, u_n) := (\tau_1 u_1, \dots, \tau_{n-1} u_{n-1}, u_n), \quad \forall \tau \in \mathbb{T}_{n-1}, \quad \forall u \in \mathbb{C}^n. \quad (3.16)$$

The corresponding moment map can be taken to be  $J = (J_1, \dots, J_{n-1}) : \mathbb{C}^n \rightarrow \mathbb{R}^{n-1}$ . Of course, the toric moment map is unique only up to a shift by an arbitrary constant.

The  $\mathbb{T}_{n-1}$ -action (3.16) and its moment map survive the symplectic reduction by the  $U(1)$ -action  $\psi$  and give rise to the so-called “rotational  $\mathbb{T}_{n-1}$ -action” on  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$ , which thus becomes a Hamiltonian toric manifold. The rotational  $\mathbb{T}_{n-1}$ -action  $\mathcal{R} : \mathbb{T}_{n-1} \times \mathbb{CP}(n-1)$  operates according to the rule  $\mathcal{R}_\tau \circ \pi_M = \pi_M \circ R_\tau$ , where  $\pi_M : \chi^{-1}(M) \rightarrow \mathbb{CP}(n-1)$  is the canonical projection. We choose its moment map to be

$$\mathcal{J} = (\mathcal{J}_1, \dots, \mathcal{J}_{n-1}) : \mathbb{CP}(n-1) \rightarrow \mathbb{R}^{n-1} \quad \text{with} \quad \mathcal{J}_k := \hat{J}_k + g. \quad (3.17)$$

Here the constant  $g$  is related to  $y$  and  $a$  as given previously (1.6). It is then easily seen that *the image of  $\mathbb{CP}(n-1)$  under the toric moment map  $\mathcal{J}$  is the same polytope  $\frac{2}{a}\mathcal{P}_y$  that features in our Theorem 3.* We can therefore use the following celebrated result.

**Delzant’s theorem [3].** *Let  $(M_1, \omega_1, \Phi_1)$  and  $(M_2, \omega_2, \Phi_2)$  be Hamiltonian toric manifolds. If the images of the moment maps  $\Phi_1(M_1)$  and  $\Phi_2(M_2)$  coincide, then there exists a torus-equivariant symplectomorphism  $\phi : M_1 \rightarrow M_2$  for which  $\Phi_1 = \Phi_2 \circ \phi$ .*

The combination of the above statements directly leads to one of the main results of [5].

**Corollary.** *All three Hamiltonian toric manifolds  $(\mathbb{CP}(n-1), M\omega_{\text{FS}}, \mathcal{J})$ ,  $(P, \hat{\omega}, \hat{\alpha})$  and  $(P, \hat{\omega}, \hat{\beta})$  are equivariantly symplectomorphic to each other.*

By applying the corollary, we have “Delzant symplectomorphisms”  $\varphi_\alpha, \varphi_\beta : \mathbb{CP}(n-1) \rightarrow P$  that are subject to

$$\varphi_\alpha^* \hat{\omega} = \varphi_\beta^* \hat{\omega} = M\omega_{\text{FS}}, \quad \varphi_\alpha^* \hat{\alpha} = \varphi_\beta^* \hat{\beta} = \mathcal{J}. \quad (3.18)$$

In fact, the symplectomorphisms  $\varphi_\beta$  and  $\varphi_\alpha$  give rise to two models of  $(P, \hat{\omega}, \hat{\alpha}, \hat{\beta})$  in such a way that in terms of model (i) the functions  $\hat{\beta}_k$  become the particle-positions and the functions  $\hat{\alpha}_k$  become the action-variables of the compactified Ruijsenaars-Schneider system, and their role is interchanged in model (ii). In both cases the model of  $(P, \hat{\omega})$  itself is provided by  $(\mathbb{C}P(n-1), M\omega_{\text{FS}})$ , which serves as the completed phase space of the  $\text{III}_b$  system equipped with the global particle-position variables  $\mathcal{J}_k$ . Before explaining these statements, we need some more preparation.

First, following [9, 5], we introduce Darboux coordinates on the dense open submanifold  $\mathbb{C}P(n-1)_0 \subset \mathbb{C}P(n-1)$  where none of the homogeneous coordinates can vanish. To do this, consider the manifold

$$\frac{2}{a}\mathcal{P}_y^0 \times \mathbb{T}_{n-1} = \{(\gamma, \tau)\}. \quad (3.19)$$

Here  $\mathcal{P}_y^0$  is the interior of the polytope (3.13) and we also write  $\tau_j = e^{\theta_j}$  ( $j = 1, \dots, n-1$ ). We define the map  $\mathcal{E} : \frac{2}{a}\mathcal{P}_y^0 \times \mathbb{T}_{n-1} \rightarrow \chi^{-1}(M)$  by requiring that  $\mathcal{E} : (\gamma, \tau) \mapsto (u_1, \dots, u_{n-1}, u_n)$  according to

$$u_j = \tau_j \sqrt{\gamma_j - g} \quad (j = 1, \dots, n-1), \quad u_n = \sqrt{M + (n-1)g - \sum_{j=1}^{n-1} \gamma_j}. \quad (3.20)$$

Then  $\pi_M \circ \mathcal{E} : \frac{2}{a}\mathcal{P}_y^0 \times \mathbb{T}_{n-1} \rightarrow \mathbb{C}P(n-1)_0$  is a diffeomorphism satisfying

$$(\pi_M \circ \mathcal{E})^*(M\omega_{\text{FS}}) = i \sum_{k=1}^{n-1} d\gamma_k \wedge d\tau_k \tau_k^{-1} = \sum_{k=1}^{n-1} d\theta_k \wedge d\gamma_k. \quad (3.21)$$

Second, we identify the local phase space  $M_y^{\text{loc}} = \mathcal{D}_y \times S\mathbb{T}_n$  (1.2) of the  $\text{III}_b$  system with  $\frac{2}{a}\mathcal{P}_y^0 \times \mathbb{T}_{n-1}$  by means of the map  $\mathcal{F} : (\gamma, \tau) \mapsto (\delta(a\gamma/2), \Theta(\tau))$  given by

$$\delta(a\gamma/2) = e^{-ia \sum_{k=1}^{n-1} \gamma_k \Lambda_k}, \quad \Theta(\tau) := e^{-i \sum_{k=1}^{n-1} \theta_k (E_{k,k} - E_{k+1,k+1})}, \quad (3.22)$$

where the same notations is used for  $\delta$  as in (3.8). The map  $\mathcal{F}$  verifies  $\mathcal{F}^* \Omega_a^{\text{loc}} = \sum_{k=1}^{n-1} d\theta_k \wedge d\gamma_k$  and its formula shows that  $\gamma$  represents the particle-positions (or rather  $(-1)$ -times the particle positions since  $\delta_j = e^{iax_j}$  was used in the Introduction) of the local  $\text{III}_b$  system. By combining the symplectic diffeomorphisms

$$M_y^{\text{loc}} \simeq \frac{2}{a}\mathcal{P}_y^0 \times \mathbb{T}_{n-1} \simeq \mathbb{C}P(n-1)_0, \quad (3.23)$$

we can identify  $M_y^{\text{loc}}$  with  $\mathbb{C}P(n-1)_0$ . Thereby the particle-positions,  $\gamma_k$ , turn into the rotational moment map,  $\mathcal{J}_k = |u_k|^2 + g$ , that remains well-defined on the whole of  $\mathbb{C}P(n-1)$ .

Now we return to the Delzant symplectomorphisms, and quote the following result from [5].

**Theorem 4.** *The Delzant symplectomorphism  $\varphi_\beta$  (3.18) can be chosen so that its restriction to  $\mathbb{C}P(n-1)_0$  operates according to the following explicit formula:*

$$\varphi_\beta(\pi_M \circ \mathcal{E}(\gamma, \tau)) = p \circ \Psi_{\eta^{-1}}(L_y^{\text{loc}}(\delta(a\gamma/2), \Theta(\tau)), \delta(a\gamma/2)), \quad (3.24)$$

where  $\eta$  is any  $U(n)$  matrix the last column of which is proportional to the vector

$$v_j(a\gamma/2, y) := \left[ \frac{\sin y}{\sin ny} \right]^{\frac{1}{2}} W_j(\delta(a\gamma/2), y), \quad j = 1, \dots, n. \quad (3.25)$$



Here the notations (1.4)-(1.5) are used, and  $\Psi_{\eta^{-1}}$  acts by componentwise conjugation (3.3). The ambiguity in the definition of  $\eta \in U(n)$  is killed by the projection map  $p : \mu^{-1}(\mu_0) \rightarrow P$ .

One of the non-trivial points of Theorem 4 is that the map given on  $\mathbb{C}P(n-1)_0$  by (3.24) extends to a *globally well-defined* symplectomorphism  $\varphi_\beta : \mathbb{C}P(n-1) \rightarrow P$ . We have not (yet) studied what is the most general global Delzant symplectomorphism, since formula (3.24) gives one and this is enough for our purpose. Namely, it follows from formula (3.24) with (3.10) that

$$\begin{aligned}\hat{\alpha}_k \circ \varphi_\beta(\pi_M \circ \mathcal{E}(\gamma, \tau)) &= \frac{2}{a} \xi_k(L_y^{\text{loc}}(\delta(a\gamma/2), \Theta(\tau))), \\ \hat{\beta}_k \circ \varphi_\beta(\pi_M \circ \mathcal{E}(\gamma, \tau)) &= \mathcal{J}_k(\pi_M \circ \mathcal{E}(\gamma, \tau)) = \gamma_k.\end{aligned}\tag{3.26}$$

This tells us that  $\varphi_\beta$  converts the reduced spectral Hamiltonians  $\hat{\alpha}$  into the action-variables of the compactified Ruijsenaars-Schneider system, whose global particle-position variables on the completed phase space  $\mathbb{C}P(n-1)$  are furnished by the function  $\mathcal{J}$ .

In the paper [5] we also gave the analogous local formula of  $\varphi_\alpha$ . It shows that the application of the pull-back  $\varphi_\alpha^*$  converts the functions  $\hat{\alpha}_k$  into the global particle-positions and the functions  $\hat{\beta}_k$  into the action-variables of the compactified Ruijsenaars-Schneider system. Then it also follows that the symplectic automorphism

$$\phi := \varphi_\alpha^{-1} \circ \varphi_\beta : \mathbb{C}P(n-1) \rightarrow \mathbb{C}P(n-1)\tag{3.27}$$

is nothing but Ruijsenaars' self-duality map for the compactified system [9] that converts the particle-positions into the action-variables, and vice versa.

We finish this section with a remark clarifying the relationship between the two Delzant symplectomorphisms  $\varphi_\beta$  and  $\varphi_\alpha$  and the involution property of the duality map  $\phi$  (3.27). For this we need to note that the reduced phase space  $P$  admits a natural anti-symplectic involution,  $\hat{m}$ . In fact,  $\hat{m}$  is induced from the anti-automorphism  $m$  of the internally fused double  $D$  operating as  $m(A, B) := (\bar{B}, \bar{A})$ , where “bar” denotes complex conjugation. Similarly,  $\mathbb{C}P(n-1)$  permits the anti-symplectic involution  $\hat{\Gamma}$  induced by the anti-symplectic involution  $\Gamma$  of the symplectic vector space  $(\mathbb{C}^n, \Omega)$ ,  $\Gamma(u_1, \dots, u_{n-1}, u_n) := (\bar{u}_{n-1}, \dots, \bar{u}_1, \bar{u}_n)$ . That is,  $\Gamma$  acts as reflection composed with complex conjugation on the first  $(n-1)$  coordinates, and  $u_n$  is chosen as special in accordance with our embedding of  $M_y^{\text{loc}}$  into  $\mathbb{C}P(n-1)$  (cf. (3.20)). Now, it can be shown that if  $\varphi_\beta$  is any Delzant symplectomorphism verifying (3.18), then

$$\varphi_\alpha := \hat{m} \circ \varphi_\beta \circ \hat{\Gamma}\tag{3.28}$$

also verifies the required properties. This implies the anti-symplectic involution property

$$(\hat{\Gamma} \circ \varphi_\alpha^{-1} \circ \varphi_\beta)^2 = \text{id}_{\mathbb{C}P(n-1)}\tag{3.29}$$

and it can be also proved that  $(\varphi_\alpha^{-1} \circ \varphi_\beta)^4 = \text{id}_{\mathbb{C}P(n-1)}$  consistently with the results of [9].

## 4 Spectra from Kähler quantization

To sum up, we saw in Section 3 that the reduced phase space  $(P, \hat{\omega})$  of the quasi-Hamiltonian reduction carries two toric moment maps defined by the two sets of functions  $\hat{\alpha}_k$  and  $\hat{\beta}_k$ . We

then exhibited symplectomorphisms between  $(P, \hat{\omega})$  and  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$  that bring either of these two moment maps into the standard rotational toric moment map on  $\mathbb{CP}(n-1)$ , yielding the correspondences

$$\hat{\alpha}_k \leftrightarrow (\hat{J}_k + g) \leftrightarrow \hat{\beta}_k. \quad (4.1)$$

$\hat{J}_k \in C^\infty(\mathbb{CP}(n-1))$  descended from  $J_k \in C^\infty(\mathbb{C}^n)$  (3.14) by ordinary symplectic reduction. According to (3.26),  $\hat{\beta}_k$  represent global analogues of the particle-positions of the compactified Ruijsenaars-Schneider system for which  $\hat{\alpha}_k$  serve as the *action-variables*.

Recall that the standard Ruijsenaars-Schneider Hamiltonians,  $\mathcal{H}_r$  ( $r = 1, \dots, n-1$ ), are the real parts of the elementary symmetric functions of the Lax matrix. It follows from equations (3.24), (3.26) and (3.8)-(3.10) that unique, globally smooth extensions of these Hamiltonians are furnished by the elementary symmetric functions of the matrix

$$\delta(a\hat{\alpha}/2) = \exp \left( -ia \sum_{k=1}^{n-1} \hat{\alpha}_k \Lambda_k \right). \quad (4.2)$$

Hence, commuting quantum operators,  $\mathcal{H}_r^{\text{op}}$ , can be defined by taking the elementary symmetric functions of

$$\delta(a\hat{\alpha}^{\text{op}}/2) = \exp \left( -ia \sum_{k=1}^{n-1} \hat{\alpha}_k^{\text{op}} \Lambda_k \right), \quad (4.3)$$

if one can construct commuting self-adjoint operators  $\hat{\alpha}_k^{\text{op}}$  corresponding to the action-variables.

Now we observe that the standard Kähler quantization of  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$  gives rise to a natural quantization of the functions  $\hat{J}_k$ , and through the symplectomorphism whereby  $\hat{\alpha}_k \leftrightarrow (\hat{J}_k + g)$  this can be used to quantize the variables  $\hat{\alpha}_k$ . By this procedure, *the joint spectrum of the resulting quantized action-variables follows immediately*. Of course, application of the analogous quantization procedure to  $\hat{\beta}_k$  leads to the same spectrum.

To develop the above observation, let us outline the quantum mechanical counterpart of the classical reduction procedure whereby we obtained  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$  from the canonical symplectic space  $(\mathbb{C}^n, \Omega)$ . For this, consider the standard holomorphic quantization of  $(\mathbb{C}^n, \Omega)$ . This engenders the commuting self-adjoint operators  $J_l^{\text{op}} = u_l \frac{\partial}{\partial u_l}$ . The joint orthonormal eigenvector basis of the operators

$$(J_1^{\text{op}}, \dots, J_n^{\text{op}}) \quad (4.4)$$

can be described as the set of the  $n$ -tuples

$$|\nu_1, \dots, \nu_n\rangle \quad \text{with any } \nu_l \in \mathbb{Z}_{\geq 0}, \quad l = 1, \dots, n. \quad (4.5)$$

In other words,  $J_l^{\text{op}}$  ( $l = 1, \dots, n$ ) are “number-operators” of independent harmonic oscillators and the  $\nu_l$  are the corresponding eigenvalues. As a holomorphic function on the phase space  $\mathbb{C}^n$ , up to a normalization constant,  $|\nu_1, \dots, \nu_n\rangle \sim \prod_{l=1}^n u_l^{\nu_l}$ .

Now we can quantize  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$  by quantum Hamiltonian reduction. In effect, this requires the imposition of the quantum analogue of the  $U(1)$  moment map constraint,

$$\sum_{k=1}^n J_k = M, \quad (4.6)$$

on the states. The reduced Hilbert space is spanned by those basis vectors  $|\nu_1, \dots, \nu_n\rangle$  for which

$$\sum_{k=1}^n \nu_k = M. \quad (4.7)$$

Hence the *quantization condition*  $M \in \mathbb{Z}_{>0}$  must be satisfied. The resulting orthonormal basis of the reduced Hilbert space can be also thought of as the set of the  $(n-1)$ -tuples

$$|\nu_1, \dots, \nu_{n-1}\rangle_{\text{red}} \quad \text{subject to} \quad \nu_k \in \mathbb{Z}_{\geq 0}, \quad \sum_{k=1}^{n-1} \nu_k \leq M. \quad (4.8)$$

Such an  $(n-1)$ -tuple represents the state

$$|\nu_1, \dots, \nu_{n-1}, M - \sum_{k=1}^{n-1} \nu_k\rangle \quad (4.9)$$

in the original Hilbert space. The procedure of quantum Hamiltonian reduction yields the commuting operators  $\hat{J}_l^{\text{op}}$  that act on the reduced Hilbert space according to

$$\hat{J}_l^{\text{op}} |\nu_1, \dots, \nu_{n-1}\rangle_{\text{red}} = \nu_l |\nu_1, \dots, \nu_{n-1}\rangle_{\text{red}}. \quad (4.10)$$

In simplest terms, the operators  $\hat{J}_l^{\text{op}}$  are just the restrictions of the operators  $J_l^{\text{op}}$  to the states verifying the constraint (4.7).

It is worth noting that the reduced Hilbert space defined above is the same as the outcome of the (geometric) Kähler quantization of  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$ . This can be verified by viewing the state (4.8) as an  $U(1)$  equivariant complex function on  $\chi^{-1}(M) \subset \mathbb{C}^n$  (cf. (3.15)). The function just alluded to is the restriction of the function (4.9), and it encodes a holomorphic section of the corresponding line bundle over  $(\mathbb{CP}(n-1), M\omega_{\text{FS}})$ . Incidentally, every Hamiltonian toric manifold can be quantized in a similar way by quantum Hamiltonian reduction, as expounded in a somewhat different language in [7].

By adopting the above Kähler quantization to the action-variables of the compactified Ruijsenaars-Schneider system, we immediately obtain from (4.1) that *the joint spectrum of the quantized action-operators  $\hat{\alpha}_k^{\text{op}}$  is given by the  $(n-1)$ -tuples*

$$(\nu_1 + g, \dots, \nu_{n-1} + g) \quad \text{subject to} \quad \nu_k \in \mathbb{Z}_{\geq 0}, \quad \sum_{k=1}^{n-1} \nu_k \leq M. \quad (4.11)$$

The joint spectrum of the action-operators is non-degenerate (all joint eigenvalues have multiplicity 1), and the eigenvalues of the commuting Hamiltonians  $\mathcal{H}_r^{\text{op}}$  can be obtained as the corresponding symmetric functions of the matrix

$$\delta(a(\nu + g\varrho)/2) = \exp \left( -ia \sum_{k=1}^{n-1} (\nu_k + g) \Lambda_k \right), \quad (4.12)$$

where  $\varrho := \sum_{k=1}^{n-1} \Lambda_k$  and we remind the convention (1.6).

The above very simple considerations reproduce the result of van Diejen and Vinet [10], who determined the joint spectrum of the commuting Hamiltonians  $\mathcal{H}_r^{\text{op}}$  by means of Schrödinger

quantization. They proceeded by making the commuting formal difference operators introduced previously by Ruijsenaars self-adjoint on a finite-dimensional Hilbert space. The Hamiltonians  $\mathcal{H}_r^{\text{op}}$  were then diagonalized by a non-trivial calculation. Their finite-dimensional Hilbert space is built on the lattice of the points  $\sum_{k=1}^{n-1}(\nu_k + g)\Lambda_k$ , with the condition in (4.8), viewed as a discretization of the global particle-position variables. Here we have explained that both this discretization and the spectra of the Hamiltonians follow from Kähler quantization. That is, Kähler quantization and Schrödinger quantization give the same result.

Having reproduced the spectral result of [10], it would be interesting to derive also their eigenfunctions and the associated quantized duality map by geometric (quantization) methods.

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